

Mathematical formulae of the all-interval tone-semitone series and the quart modes

Grigori G. Amosov ^{*}and Sergei V. Golubkov [†]

Department of Higher Mathematics
Moscow Institute of Physics and Technology
141700 Dolgoprudny
RUSSIA

Department of Composition
Moscow Tchaikovsky Conservatory
125009 Moscow
RUSSIA

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Abstract

Using the residue class rings we analyze the structure of sequences of musical intervals for the all-interval tone-semitone series and the quart modes as well. The relations we stated show the enharmonic return of the series to the initial point.

1 Introduction

It seems that for the first time the musical intervals were considered as mathematical transformations of the residue class ring modulo 12 in the paper by

^{*}E-mail:gramos@mail.ru

[†]E-mail:sgolubk@mail.ru

french scientist Camille Durutte in 1855. Notice that the modular arithmetics (the residue class rings) was introduced by the famous German mathematician Carl Friedrich Gauss in 1801. Such the point of view acquired a special urgency in the XX century because the appearance of the serial techniques and the set theory especially brightly developed in the works by Milton Babbitt ([1]). It is connected with the fruitful applications of such an approach to constructing the sequencies of sounds by means of the certain group of transformations of the ring $\mathbb{Z}/12\mathbb{Z}$ possessing some mathematical symmetry.

In our sketch we shall consider from this point of view the all-interval tone-semitone series (ATsS). The scientific analysis of this object was given by one of the authors of the present paper ([3, 4]) and Irina Severina ([7]). We shall compare ATsS with the quart (non-octave) modes including the ancient Russian "custom mode", his modern prolonged version introduced by Yuri Butsko in Polyphonian Concert (1969) and the artificial "synagogue mode" (really, the custom mode being transformed) introduced by Alfred Shnitke in his 4th symphony (1984). Notice that the phenomenon of ATsS was of interest for the other researchers in different contexts ([9, 10, 6, 8]).

2 The all-interval tone-semitone series

Let us enumerate a sequence of sounds of the imaginary "infinite" keyboard by integer numbers $n = 0, \pm 1, \pm 2, \dots$, such that each two neighboring numbers are corresponding to the sounds differing by a semitone. The musical interval of a duration m can be considered as the operation transforming the set of integer numbers by the rule $n \rightarrow n + m$. Because the feature of a human ear the tones corresponding to the numbers n and $n + 12$ will be perceived as sounding in unison. Adding or subtracting from each a number n the appropriate integer number divisible by 12 it is possible to transpose all the sounds of the keyboard to one octave corresponding to the integer numbers $n = 0, 1, 2, \dots, 11$.

Denote by $\mathbb{Z}/12\mathbb{Z} = \{0, 1, 2, \dots, 11\}$ the set of sounds of this octave. If we take into account a value of interval up to adding an integer number of octaves, then the interval is a set of 11 transformations of the elements of the set $\mathbb{Z}/12\mathbb{Z}$ determined by a mathematical operation of the "adding modulo 12". This operation can be defined as follows. The result of the adding of the numbers n and m modulo 12 is the number $n + m + 12k$ (the notation is $m + n \bmod 12$), where the integer k is picked up such that

$m+n+12k \in \mathbb{Z}/12\mathbb{Z}$. Analogously one can define the "multiplication modulo 12". The result of this operation is the number $mn + 12k$, where k is chosen such that $mn + 12k \in \mathbb{Z}/12\mathbb{Z}$. The set $\mathbb{Z}/12\mathbb{Z}$ with the operations of the adding modulo 12 and the multiplication modulo 12 is said to be the ring of residue classes modulo 12.

Thus, to obtain a certain row of sounds (in which we shall take into account not only what sounds are included in it but the order of their appearance in the row also) one should define an operation T on the ring $\mathbb{Z}/12\mathbb{Z}$. In the author's papers [2, 3] ATsS was defined (with a possibility to continue to the "infinite" keyboard) such that the n -th sound of the row differs from the first on $\frac{n(n-1)}{2}$ semitones, that is the 12th sound differs from the first on 66 semitones. Remember that ATsS consists of 12 sounds because under "all intervals" of its basic form occupying five and half octaves (66:12) we mean the intervals from 1 to 11 contained inside the octave and having a musical structural sense. In other cases, it is possible to talk about all the intervals from 1 to 6 (because the intervals 6-11 are their conversions inside the octave), from 0 to 24 or the imaginary "infinite" all-interval row (within the limits of audibility and besides it). It should be especially mentioned that the intervals from 0 to 12 (24,36,...) are unisons (repetitions and octave doublings of the sounds) and they have nor musical structural but only mathematical sense. Continuing the row we get that the intervals from 13 to 23 gives us a tritone transposition of the basic form of ATsS which identically coincides with its retrograde inversion, the intervals from 25 to 35 is a repetition of sounds in the basic form and so on.

Denote $|n\rangle$ the element of the ring $\mathbb{Z}/12\mathbb{Z}$ corresponding to the $(n-1)$ th sound of ATsS. Then, the operation V determining the series by the formula $|n\rangle = Vn$, $n = 0, 1, 2, \dots$ can be defined by means of the following recurring formula,

$$V0 = 0, \quad Vn = |n-1\rangle + n \bmod 12, \quad n = 1, 2, 3, \dots \quad (1)$$

The solution to (1) is given by the formula of the arithmetic progression:

$$|n\rangle = \frac{n(n+1)}{2} \bmod 12, \quad n = 0, 1, 2, \dots \quad (2)$$

Notice that it follows from (2) that

$$|n+12\rangle = |n\rangle + 6 \bmod 12. \quad (3)$$

At first, it implies that $|n + 24k| \geq |n|$, $k = 1, 2, \dots$, i.e. the basic form of ATsS generates a series of 24 elements of the ring $\mathbb{Z}/12\mathbb{Z}$ being repeated cyclically with the period equal to 24. On the other hand, the operation T determined on the ring $\mathbb{Z}/12\mathbb{Z}$ by the formula

$$Tn = n + 6 \text{ mod } 12$$

is a tritone transposition from the musical point of view. Thus, (3) implies that all the sounds from 13 to 24 obtained by a subsequent adding to the basic form of ATsS the intervals increasing by means of the same principle are tritone transpositions (coinciding with retrograde inversions) of the first 12 sounds. Notice that the continued in two times ("doubled") form of ATsS returns enharmonically to the initial point after 17.5 octaves (210:12) of the imaginary "infinite" keyboard and consists of 24 sounds (12x2).

Then, the operation R_m defined on the row x consisting of m elements $[x_1, \dots, x_m]$ of the ring $\mathbb{Z}/12\mathbb{Z}$ by the formula

$$R_m x_s = x_{m-s+1}$$

results in a retrograde inversion of the initial row. It follows from (2) that

$$R_6|n| = |n| + 6n + 3 \text{ mod } 12. \quad (4)$$

On the other hands, the same formula implies that

$$|n + 6| = |n| + 6n + 9 \text{ mod } 12. \quad (5)$$

Thus, formulae (4) and (5) give us the relation

$$|n + 6| = R_6|n| + 6 \text{ mod } 12$$

confirming that in the basic form of ATsS the sounds 7 – 12 are obtained by means of the tritone transposition of the retrograde inversion of the sounds 1 – 6.

Notice that the ring $\mathbb{Z}/12\mathbb{Z}$ includes zero divisors, i.e. the non-zero elements m and n such that $mn = 0 \text{ mod } 12$. Namely, there are two such the elements, that are 3 and 4 because

$$3 \cdot 4 = 12 = 0 \text{ mod } 12.$$

The existence of the zero divisors determines a reiteration of sounds in the series of 12 elements of the basic form of ATsS. In fact, the identity

$$|n\rangle = |m\rangle$$

is equivalent to

$$(n - m)(n + m + 1) = 24k$$

for a certain integer number k . Given integer numbers m and n , one of the numbers $n - m$ or $n + m + 1$ is even and the other is odd. If $n - m$ is even, then we get the product of two integer numbers of the form $\frac{n-m}{2}(n+m+1) = 12k$, and the zero divisors in $\mathbb{Z}/12\mathbb{Z}$ are $\frac{n-m}{2}$ and $n+m+1$. Analogously, if $n+m+1$ is even, then $(n - m)\frac{n+m+1}{2} = 12k$ is a product of two integer numbers, and the zero divisors are $n - m$ and $\frac{n+m+1}{2}$. Substituting as zero divisors the pair (3, 4) and the pair (1, 12) as well, we obtain that 12 sounds of the basic form of ATsS contain 4 pairs of reiterated sounds (remark that although 12 is not a zero divisor but it is zero in the ring $\mathbb{Z}/12\mathbb{Z}$).

So, 12 sounds of the basic form of ATsS contain 4 pairs of reiterated sounds and 4 non-reiterated sounds, that are 8 different sound pitches at all. For the whole chromatic scale one needs 4 sounds more. Why? Let us consider the equation

$$|n\rangle = m \bmod 12. \quad (6)$$

The equation (6) in the ring $\mathbb{Z}/12\mathbb{Z}$ is equivalent to the condition

$$n^2 + n = 2m + 24k. \quad (7)$$

The number on the right hand side of the equality (7) is always even but the number in the left hand side is even only if n is even or $n = 1$ or $n = 11$. Hence, the equation (6) is solvable only for 8 different values of m .

To obtain failing 4 sounds one can use the inversion I acting to the elements of the ring $\mathbb{Z}/12\mathbb{Z}$ by the formula

$$In = 12 - n \bmod 12.$$

Following down from the first sound of the basic form of ATsS let us construct 12 sounds of its inversion form by the formula

$$|-n\rangle = I|n\rangle, \quad n = 0, 1, \dots, 11.$$

Then, the equation $|-n\rangle = m \bmod 12$ is equivalent to the condition

$$n^2 + n = 2(12 - m) + 24k. \quad (8)$$

The formula (8) as well as the formula (7) is solvable only in 8 cases but the solutions to (7) are $m = 0, 1, 3, 4, 6, 7, 9, 10$ while the solutions to (8) are $Im = 0, 2, 3, 5, 6, 8, 9, 11$ which include the failing numbers $m = 2, 5, 8, 11$.

3 The quart modes

Now let us proceed to the mathematical formulae of the ancient Russian "custom mode" and the artificial "synagogue mode" being constructed by quarts which are filled by seconds (the synagogue mode was introduced by Alfred Shnitke in his 4th symphony ([2, 5])). In particular, it will be interesting for us to compare them with the formulae of ATsS obtained above. The interval structure of the custom mode is the following

$$(5 =)2 + 2 + 1, 2 + 2 + 1, 2 + 2 + 1, \dots (\text{large}), \text{ or}$$

$$(5 =)2 + 1 + 2, 2 + 1 + 2, 2 + 1 + 2, \dots (\text{small}), \text{ or}$$

$$(5 =)1 + 2 + 2, 1 + 2 + 2, 1 + 2 + 2, \dots (\text{reduced}),$$

and for the "synagogue mode" is

$$(5 =)1 + 1 + 3, 1 + 1 + 3, 1 + 1 + 3, \dots, \text{ or}$$

$$(5 =)1 + 3 + 1, 1 + 3 + 1, 1 + 3 + 1, \dots, \text{ or}$$

$$(5 =)3 + 1 + 1, 3 + 1 + 1, 3 + 1 + 1, \dots$$

Denote $|n >$ the series of the elements of the ring $\mathbb{Z}/12\mathbb{Z}$ generated by the interval rows of these modes. In the both cases, we get

$$|n + 3 > = |n > + 5 \bmod 12. \quad (9)$$

The least common multiple of the numbers 5 and 12 is $60 = 5 \cdot 12$. In this way, it follows from (9) that the continued quart modes enharmonically return to "the initial point" through 5 octaves ($60 : 12$) and consist of 36 sounds ($3 \cdot 12$). The formula (9) shows a mathematical symmetry in the construction of the interval rows considered as well as the formula (3) determines a regularity of the enharmonic return to "the initial point" on the imaginary "infinite" keyboard of the doubled form of ATsS.

To investigate the quart modes it is useful to involve the ring $\mathbb{Z}/5\mathbb{Z} = \{0, 1, 2, 3, 4\}$ in which, analogously to the case of the ring $\mathbb{Z}/12\mathbb{Z}$, the operations of the "adding modulo 5" and the "multiplication modulo 5" are

introduced. Notice that the number 5 (unlike 12) is prime. Hence the ring $\mathbb{Z}/5\mathbb{Z}$ doesn't contain zero divisors. It follows that $\mathbb{Z}/5\mathbb{Z}$ is a field (in spite of the operations of adding, subtraction and multiplication the operation of division is defined).

Let us denote the series of elements of the field $\mathbb{Z}/5\mathbb{Z}$ generated by the interval rows of the continued (large, small and reduced) custom and synagogue modes in each of the cases considered above by $|L_n>$, $|S_n>$, $|R_n>$ and $|SG_n>$, respectively. In all the cases we obtain three elements of the field $\mathbb{Z}/5\mathbb{Z}$, namely,

$$\begin{aligned} |L_0> &= 0, \quad |L_1> = 2, \quad |L_2> = 4, \\ |S_0> &= 0, \quad |S_1> = 2, \quad |S_2> = 3, \\ |R_0> &= 0, \quad |R_1> = 1, \quad |R_2> = 3, \\ |SG_0> &= 0, \quad |SG_1> = 1, \quad |SG_2> = 2, \quad or \\ |SG_0> &= 0, \quad |SG_1> = 1, \quad |SG_2> = 4, \quad or \\ |SG_0> &= 0, \quad |SG_1> = 3, \quad |SG_2> = 4. \end{aligned}$$

It is easy to see that the sequences considered consisting of triples of the elements of the field $\mathbb{Z}/5\mathbb{Z}$ use up all possible constructions which can be generated by the quart modes determined by the formula (9). Notice that from the viewpoint of mathematics the large custom mode possesses the most common symmetry because the elements of the field $\mathbb{Z}/5\mathbb{Z}$ which are generated by it form a cyclic group with the generator 2:

$$|L_n> = 2n \text{ mod } 5, \quad n = 0, 1, 2.$$

4 Discussions

We have derived the basic mathematical formulae of the all-interval tone-semitone series and the quart modes including the ancient Russian custom mode and the artificial "synagogue mode" proposed by Alfred Shnitke as well. The information contained in these formulae allows to know when the row will enharmonically return to the "initial point" and what sounds does it consist of also. The musical sense of some mathematical operations on the residue rings is revealed.

References

- [1] Andreatta M. (2003) *Methodes algebriques en musique et musicologie du XX siecle: aspects theoriques, analytiques et compositionnels*. PhD Thesis. Paris: IRCAM. (French)
- [2] Golubkov S.V. (1999) On the question of the "synagogue mode" (to the memory of Alfred Shnitke). *Musical Academy*, 2, 81-84. (Russian)
- [3] Golubkov S.V. (1995) The composition techniques based on the all-interval series: a synthesis of the seriality and the modality. *Musical art of the XX century: the collection of papers*, 2 (Ed. T.Dubravskaya). Moscow, 140-171. (Russian)
- [4] Golubkov S.V. (1994) The synthesis of the dodecaphony and the modality in the structure of the all-interval tone-semitone series. *Musical Academy*, 5, 126-132. (Russian)
- [5] Holopova V., Chigareva E. (1990) Alfred Shnitke. Moscow. (Russian)
- [6] Kulibaeva M. (2005) MS Thesis. The Moscow Tchaikovsky Conservatory. (Russian)
- [7] Severina I. (2001) *The modal-serial systems in the native music of the XX century: the syncretism and the synthesis of organization principals*. PhD Thesis. Moscow: The Gnesin Musical Academy. (Russian)
- [8] Sharova M. (2001) *The individual syntax as a phenomenon of the composer practice*. MS Thesis: The Moscow Tchaikovsky Conservatory. (Russian)
- [9] Voinova M. (2002) The organ music of the young Moscow composers. *To centenary anniversary of the organ in Bolshoi Zal of Moscow Tchaikovsky Conservatory: the collection of papers* (Ed. M. Voinova). Moscow, 79-86. (Russian)
- [10] Voinova M. (2003) *Problems of the modern organ music*. PhD Thesis. The Moscow Tchaikovsky Conservatory. (Russian)